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ON BUCKLING OF AXISYMMETRIC THIN ELASTIC-PLASTIC SHELLS

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Abstract—A buckling criterion for shells with an axisymmetric middle surface and subjected to edge loads and hydrostatic surface pressure loading is formulated starting from Hill's three-dimensional continuum theory for uniqueness of deformation of inelastic solids. It turns out that a physically consistent two-dimensional set of equations may be derived for a quite general class of strain-hardening elastic-plastic solids, the only essential restriction being that of a smooth yield function. The intrinsic errors inherent in the derived rate equations, being an integral part of an eigenvalue problem, are discussed in relation to a circular cylinder under axial compression. Analytical results are given which illustrate the influence of the constitutive properties and the boundary contraints on the magnitude of the critical load.

INTRODUCTION

The occurrence of bifurcations at loading of shells exhibiting elastic-plastic behaviour has received increasing attention during past years. Suffice it to mention in this context, among other important contributions, the analytical cylinder studies by Batterman[1, 2] and Ariaratnam and Dubey[3] together with the recent more general numerical approach by Bushnell[4]. Through the method worked out by Bushnell buckling problems may be successfully solved for perfect shells of revolution in which during steady loading the principal stress directions coincide with the meridional and longitudinal directions, the material behaviour then being described by aid of an isotropic J_2-theory, with an option for a free choice of the 'in-plane' shear modulus though. Among other features such limitations are not inherent in the present analytical attempt.

When dealing with inelastic solids boundary value problems must necessarily be set as rate problems due to the deformation history dependence of the instantaneous mechanical material properties. Instead of pursuing the direct method of deriving rate equations based solely on the kinematics of the middle surface through a common shell approach, in the present case Hill's three-dimensional continuum theory for uniqueness of deformation has been adopted as a starting-point. The rate equations and associated natural boundary conditions may then be derived from a three-dimensional variational principle valid for particular solids. An obvious advantage of this approach, when dealing with complicated problems, is the proved existence of an associated Rayleigh principle which under some circumstances provides the possibility of obtaining upper bounds for critical load parameters.

The derivation of a variational principle applicable to buckling of thin shells from a three-dimensional theory is by no means an automatic procedure which is recognized by anyone familiar with the line of progress of elastic shell theory. Through fundamental studies by a number of distinguished writers, summarized, e.g. in [5], theories now exist which are consistent for finite deformations of thin elastic shells. By introducing middle surface strains and curvature changes as basic and sole kinematical variables, equilibrium equations and dynamic boundary conditions for conjugate stress resultants and stress couples may be derived by aid of the principle of virtual work. The equations generated from these theories may be regarded as exact in an asymptotic sense at least for shells with constitutive properties which are homogeneous through the shell wall and of plane isotropy.

The situation is more complicated for a shell in a strain-hardening plastic state and particularly when the fundamental stress state is not a membrane state and the shell material hardening properties are not even locally homogeneous. It remains in doubt still whether the introduction of common, in some sense simple, kinematical assumptions will yield results correct to first order in the thin-shell limit. There exists, however, a class of practically important problems where the variation of curvature is smooth and the boundary conditions are such that essentially a membrane stress state prevails in the steady state. The search for bifurcations in such situations then reduces to the solution of a two-dimensional hypo-elastic (anisotropic) rate problem.

Hill's theorems have been earlier applied with success mostly to beam and plate problems and in the former case it proved possible to account for the effect of shear stiffening when investigating the buckling load of an elastic-plastic column [6]. It would seem of interest therefore to investigate the structure of the shell equations generated by means of a simple two-dimensional specialization retaining some degree of generality. A corresponding success as in plate problems may not be taken for granted though.

GENERAL BIFURCATION THEORY

During a substantial time period starting in the middle of the fifties many aspects of the rate problem occurring at finite deformation of solids of quite general material properties and subjected to a diversity of boundary conditions have been studied by Hill. These efforts have resulted in powerful extremum and variational principles and uniqueness and stability criteria. In a recent review in parts by Sewell [7] a complete bibliography has been listed and generalizations offered for the case of singular yield functions. Only a short account of the main results applicable in the present context is given here.

When formulating the boundary value rate problem it is assumed that all relevant details regarding the mechanical state is known at a generic stage of the deformation process. It is then convenient to choose as a reference configuration the one prevailing when the continuum is investigated for uniqueness of continued quasi-static motion. By aid of the contravariant components s^{ij} of the nominal (first Piola–Kirchhoff) stress tensor the equilibrium conditions for continued motion takes the simple form

$$\dot{s}^{ij}{}_{,i} = 0 \tag{1}$$

referred to convected coordinates. A superposed dot denotes material derivation with respect to any parameter t (say), which increases monotonically with time and a comma denotes covariant derivation with respect to the metric in the reference configuration.

The nominal stress-rate is then related to the more familiar Cauchy stress-rate through the standard formula

$$\dot{s}^{ij} = \dot{\sigma}^{ij} + \sigma^{ij} v^{k}{}_{,k} - \sigma^{jk} v^{i}{}_{,k} \tag{2}$$

where v_i are particle velocities.

An appropriate formulation of the constitutive properties of strain-hardening elastic-plastic solids is a matter permanently subjected to theoretical and experimental studies. Fortunately, at least from a formal view-point, for the present purpose only the instantaneous material behaviour is of interest and it is sufficient to assume that at a generic instant, whatever its general form may be, the constitutive equation may be expressed in the classical rate form

$$d_{ij} = \left(M_{ijkl} + \frac{\delta}{h}\mu_{ij}\mu_{kl}\right)\dot{\tau}^{kl} \tag{3}$$

to describe continued deformation.

The strain-rate measure chosen in (3) is the convected derivative of Green's strain tensor which, with the present choice of reference configuration, may be referred to particle velocities v_i through

$$2d_{ij} = v_{i,j} + v_{j,i} \tag{4}$$

The elastic compliance tensor M_{ijkl} in (3) is left unspecified except for the assumption of $ij \leftrightarrow kl$ symmetry (apart from the ordinary symmetries) and the scalar δ equals one when plastic flow occurs and zero otherwise. The unique unit normal μ_{ij} to the yield surface in stress space and the scalar measure h of the rate of strain-hardening at a generic instant are in general functionals of the deformation history.

The conjugate stress-rate measure introduced in (3), being the convected rate of the contravariant symmetric Kirchhoff (second Piola–Kirchhoff) stress, does not lend itself easily to a physical interpretation. It is related to the nominal stress rate, though, through

$$\dot{\tau}^{ij} = \dot{s}^{ij} - \sigma^{ik} v^{j}{}_{,k} \tag{5}$$

which, due to the existing symmetries, leads to a convenient consequence; a potential

$$U = \tfrac{1}{2}(\dot{\tau}^{ij} d_{ij} + \sigma^{ij} v^{k}{}_{,i} v_{k,j}), \tag{6}$$

being a quadratic function of velocity gradients through (3) and generating

$$\dot{s}^{ij} = \frac{\partial U}{\partial v_{j,i}}, \tag{7}$$

may be introduced. This feature has far-reaching consequences when it comes to the solution of standard boundary value problems.

In a general situation, when determining the 'tangent moduli' appearing in (3), the definitions of particular stress rates must be distinguished. With the present application in mind this matter might not be crucial as metal structures ordinarily buckle in modes with rotations dominating over strains and diverse commonly adopted objective stress-rate measures differ only through terms of type stress times strain-rate. When 'stretching' effects are important this issue needs due consideration though as detailed by Hill[8].

In a standard boundary value problem when the nominal traction-rate $\dot{F}_i$ is prescribed on part of the body surface S and particle velocities on the remainder, an obviously sufficient condition for uniqueness of continued deformation under an infinitesimal change of the boundary conditions, is

$$I = \int \Delta \dot{F}^{j} \Delta v_j \, \mathrm{d}S \neq 0 \tag{8}$$

where Δ denotes the difference between two solutions to the field variable following and

$$\dot{F}^{j} = l_i \dot{s}^{ij} \tag{9}$$

l_i being the current outward unit normal to the boundary surface.

From stability aspects only positive values of I are of interest and by aid of the divergence theorem and (1) it may be transformed to a volume integral

$$I = \int \Delta \dot{s}^{ij} \Delta v_{j,i} \, \mathrm{d}V > 0 \tag{10}$$

or by aid of (7)

$$I = \int \Delta \frac{\partial U}{\partial v_{j,i}} \Delta v_{j,i} \, \mathrm{d}V > 0. \tag{11}$$

It is evident from the appearance of (8) that (11) is applicable also for boundary conditions of mixed traction-rate/particle velocity type.

To investigate the sign of I in a general situation does in nontrivial situations pose an awkward problem due to the non-linearity of the constitutive equation for material elements at yield. It was shown by Hill[9], however, that for common elastic-plastic solids a more practical though weakened form of (11) may be obtained by adopting a potential U_L being constructed from the plastic loading branch of (3) for material elements at yield. For such a 'linear comparison solid', which formally has the same properties as an anisotropic hypo-elastic solid in Truesdell's terminology, (11) reduces to

$$I_L = \int \frac{\partial U_L}{\partial v_{j,i}} v_{j,i} \, \mathrm{d}V > 0 \tag{12}$$

as U_L is then a single-valued function of the difference between two velocity gradient fields, the Δ-symbol being deleted for convenience.

Further by Euler's theorem for homogeneous functions

$$I_L = 2 \int U_L \, \mathrm{d}V > 0 \tag{13}$$

In many practical situations, incidentally all the ones analysed so far in the present spirit and to the writer's knowledge, the considered bifurcation modes involve no unloading in plastic sense implying that inequalities (11) and (13) are of equal strength.

In essence the bifurcation problem now reduces to the search for eigensolutions to be superposed on the known steady state solution. As was shown by Hill [9, 10] variational methods are then available and eigenmodes may be found from

$$\delta I_L = 0 \tag{14}$$

with the field variables (actually formed then from the difference between two fields) subjected to homogeneous boundary conditions. The amplitudes of the eigenmodes are then bounded from above through the constitutive restriction of continued plastic loading in regions at yield. It has been assumed throughout that (3) has a unique inverse and that $\dot{\tau}^{ij} d_{ij}$ is positive definite so that incipient quasistatic motion is unique.

When for instance hydrostatic pressure loading p is acting on part of the boundary surface S_p, which is a particular case of a family of self-adjoint boundary conditions studied by Hill [11], (14) should be supplemented to read

$$\delta\left(\int U_L \, \mathrm{d}V + \int p(l_i v^k{}_{,k} - l_j v^j{}_{,i}) v^i \, \mathrm{d}S_p\right) = 0 \tag{15}$$

A BUCKLING CRITERION FOR SHELLS HAVING AN AXISYMMETRIC MIDDLE SURFACE

When composing the introduced functional I_L for a thin-walled shell in a membrane state, it seems justified to assume that continued motion will occur under approximately plane transverse stress when the transverse shear moduli are of the same order as the inplane moduli. This admits the relevant part of the constitutive eqn (3) to be expressed (by aid of physical components) in a reduced form

$$\left.\begin{aligned} \dot{\tau}_{11} &= a_{11} d_{11} + a_{12} d_{22} + 2a_{13} d_{12} \\ \dot{\tau}_{22} &= a_{12} d_{11} + a_{22} d_{22} + 2a_{23} d_{12} \\ \dot{\tau}_{12} &= a_{13} d_{11} + a_{23} d_{22} + 2a_{33} d_{12} \end{aligned}\right\}. \tag{16}$$

As regards the kinematics the Kirchhoff–Love assumptions are utilized implying that material elements normal to the reference middle surface remain normal to the deformed middle surface and do not change their length. In this spirit relevant buckling loads are expected to be well approximated by a particle velocity field (essentially conserving normals)

$$v_\alpha = u_\alpha - z w_{,\alpha} \quad (\alpha = 1, 2), \quad v_3 = w \tag{17}$$

where the in-plane middle surface velocity components u_α are arbitrary functions of orthogonal middle surface coordinates x^β, w is the (locally) constant velocity in the normal direction and z denotes the distance of an arbitrary particle from the middle surface.

The assumption of vanishing transverse normal strain-rate is in general in contradiction at least to the assumption of a continued transverse plane stress state. It is possible to remedy this shortcoming by supplementing the expression for the transverse particle velocity w in (17) through terms which yield continued vanishing transverse normal stress. When transverse shearing effects are unimportant the same result may be obtained for the present purpose, in a customary and perhaps more simple way, by deriving the plane stress form of the constitutive equation, as in (16), before the potential U_L is formed and appropriate velocity gradients introduced.

At this instant it proves convenient, in case of an axisymmetric shell middle surface, to introduce the following notation; θ is the angle between a normal to the middle surface of the shell and the symmetry axis, ϕ is a polar angle, (u, v, w) are the physical velocity components of particles on the middle surface having principal radii R_1, R_2 and consequently polar radius $r = R_2 \sin\theta$. The operators $\partial/\partial x = R_1^{-1}\partial/\partial\theta$ and $\partial/\partial y = r^{-1}\partial/\partial\phi$ (which are not commutative in general as for compatibility reasons $\partial r/\partial x = \cos\theta$) are symbolized by superscripts prime and dot† respectively.

Then by aid of (15), (16) and (17) for a steady membrane stress state $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$

$$
\begin{aligned}
I_L = \int \Big(& (a_{11}+\sigma_{xx})[u' + w/R_1 - z(w' - u/R_1)']^2 + 2a_{12}[u' + w/R_1 - z(w' - u/R_1)'] \\
& \times [u\cos\theta/r + \dot{v} + w\sin\theta/r - z(\ddot{w} + w'\cos\theta/r)] + 2a_{13}[u' + w/R_1 - z(w' - u/R_1)'] \\
& \times [\dot{u} + v' - v\cos\theta/r - z(2\dot{w} - v\sin\theta/r)'] + (a_{22} + \sigma_{yy}) \\
& \times [u\cos\theta/r + \dot{v} + w\sin\theta/r - z(\ddot{w} + w'\cos\theta/r)]^2 + 2a_{23} \\
& \times [u\cos\theta/r + \dot{v} + w\sin\theta/r - z(\ddot{w} + w'\cos\theta/r)][\dot{u} + v' - v\cos\theta/r - z(2\dot{w} - v\sin\theta/r)'] \\
& + a_{33}[\dot{u} + v' - v\cos\theta/r - z(2\dot{w} - v\sin\theta/r)']^2 + \sigma_{xx} \\
& \times \{[v' - z(\dot{w} - v\sin\theta/r)']^2 + (w' - u/R_1)^2\} + 2\sigma_{xy}\{[u' + w/R_1 - z(w' - u/R_1)'] \\
& \times (\dot{u} - v\cos\theta/r - z\dot{w}') + [v' - z(\dot{w} - v\sin\theta/r)'] \\
& \times [u\cos\theta/r + v' + w\sin\theta/r - z(\ddot{w} + w'\cos\theta/r)] + (w' - u/R_1)(\dot{w} - v\sin\theta/r)\} \\
& + \sigma_{yy}[(\dot{u} - v\cos\theta/r - z\dot{w}')^2 + (\dot{w} - v\sin\theta/r)^2] \Big)\,\mathrm{d}V \\
& + \int p[(u\cos\theta/r + u' + \dot{v} + w/R_1 + w\sin\theta/r)w \\
& - (w' - u/R_1)u - (\dot{w} - v\sin\theta/r)v]\,\mathrm{d}S_p
\end{aligned}
\tag{18}
$$

where the surface integral should be evaluated at the middle surface.

Guided by the results of Koiter's[12] thorough discussion of the relative errors introduced through the approximative character of the basic assumptions in elastic shell theory, terms of the order $|z|/R_1$ and $|z|/R_2$ have been deleted in (18) in expressions accounting for the change of curvature of the middle surface whenever compared to unity. At the present state of knowledge there seems to be no justification to refine the kinematics of the situation further than this although it is clear that the resulting approximation rules out buckling modes in which the main rotation occurs around a normal to the middle surface. Such modes seem unlikely to appear though in axisymmetric structures which deserve to be labeled shells in any sense.

Integrating the volume integral in (18) in the thickness direction utilizing the metric of the middle surface and introducing the notation $A_{ij} = ta_{ij}$, $n_{\alpha\beta} = t\sigma_{\alpha\beta}$, $K = t^2/12$ (shell wall thickness t, not necessarily homogeneous but varying to a small amount though), $\rho = r/R_1$, application of (15) yields the Euler equations

†Not to be confused with the earlier introduced symbol for material derivation.

$$
\left.\begin{aligned}
&[D_x(A_{11}+n_{xx})r - A_{12}\cos\theta + rD_y(A_{13}+n_{xy})](u' + w/R_1) - \frac{1}{R_1}[D_x(A_{11}+n_{xx})rKD_x \\
&\quad - n_{xx}r - prR_1](w' - u/R_1) + [D_xA_{12}r - (A_{22}+n_{yy})\cos\theta + rD_yA_{23}] \\
&\quad \times(u\cos\theta/r + v^{\cdot} + w\sin\theta/r) - \rho D_xA_{12}K(w^{\cdot\cdot} + w'\cos\theta/r) + (D_xA_{13}r \\
&\quad - A_{23}\cos\theta + rD_yA_{33})(u^{\cdot} + v' - v\cos\theta/r) - \frac{1}{R_1}D_xA_{13}KrD_x(2w^{\cdot} - v\sin\theta/r) \\
&\quad + (D_xn_{xy}r + rD_yn_{yy})(u^{\cdot} - v\cos\theta/r) - n_{xy}[v'\cos\theta - \rho(w^{\cdot} - v\sin\theta/r)] \\
&\quad - \frac{1}{R_1}D_xn_{xy}Krw^{\cdot\prime} = 0 \\
&[rD_yA_{12} + (A_{13}+n_{xy})\cos\theta + D_xA_{13}r](u' + w/R_1) - \sin\theta\left(\frac{1}{r}D_xA_{13}KrD_x - n_{xy}\right) \\
&\quad \times(w' - u/R_1) + [rD_y(A_{22}+n_{yy}) + A_{23}\cos\theta + D_x(A_{23}+n_{xy})r] \\
&\quad \times(u\cos\theta/r + v^{\cdot} + w\sin\theta/r) + (rD_yA_{23} + A_{33}\cos\theta + D_xA_{33}r)(u^{\cdot} + v' - v\cos\theta/r) \\
&\quad - \frac{\sin\theta}{r}D_x(A_{23}+n_{xy})Kr(w^{\cdot\cdot} + w'\cos\theta/r) \\
&\quad - \frac{\sin\theta}{r}D_xA_{33}KrD_x(2w^{\cdot} - v\sin\theta/r) + (D_xn_{xx}r + rD_yn_{xy})v' - \sin\theta \\
&\quad \times\left(\frac{1}{r}D_xn_{xx}KrD_x - n_{yy} - pr/\sin\theta\right)(w^{\cdot} - v\sin\theta/r) + n_{yy}\cos\theta(u^{\cdot} - v\cos\theta/r) = 0 \\
&[\rho(A_{11}+n_{xx}) + A_{12}\sin\theta + pr](u' + w/R_1) + [D_{xx}(A_{11}+n_{xx})KrD_x - D_xA_{12}K\cos\theta D_x \\
&\quad + rD_{yy}A_{12}KD_x + 2D_{xy}A_{13}KrD_x - D_xn_{xx}r - rD_yn_{xy} + D_{xy}n_{xy}KrD_x](w' - u/R_1) \\
&\quad + [D_{xx}A_{12}Kr - D_x(A_{22}+n_{yy})K\cos\theta + rD_{yy}(A_{22}+n_{yy})K + rD_{xy}(2A_{23}+n_{xy})K] \\
&\quad \times(w^{\cdot\cdot} + w'\cos\theta/r) + [\rho A_{12} + (A_{22}+n_{yy})\sin\theta + pr](u\cos\theta/r + v^{\cdot} + w\sin\theta/r) \\
&\quad + (\rho A_{13} + A_{23}\sin\theta)(u^{\cdot} + v' - v\cos\theta/r) + (D_{xx}A_{13}KrD_x \\
&\quad - D_xA_{23}K\cos\theta D_x + rD_{yy}A_{23}KD_x + 2D_{xy}A_{33}KrD_x)(2w^{\cdot} - v\sin\theta/r) \\
&\quad + [D_{xy}n_{xx}KrD_x - D_xn_{xy}(r + K\cos\theta D_x) + rD_{yy}n_{xy}KD_x - rD_yn_{yy}](w^{\cdot} - v\sin\theta/r) \\
&\quad + n_{xy}[\rho[u^{\cdot} - v\cos\theta/r) + v'\sin\theta] + (D_{xx}n_{xy}Kr + D_{xy}n_{yy}Kr)w^{\cdot\prime} = 0
\end{aligned}\right\} \tag{19}
$$

and the associated boundary condition

$$
\begin{aligned}
&\oint\Bigg(\Bigg\{(A_{11}+n_{xx}\left[u' + w/R_1 - \frac{K}{R_1}(w' - u/R_1)'\right] + A_{12}\Big[u\cos\theta/r + v^{\cdot} + w\sin\theta/r \\
&- \frac{K}{R_1}(w^{\cdot\cdot} + w'\cos\theta/r)\Big] + A_{13}\left[u^{\cdot} + v' - v\cos\theta/r - \frac{K}{R_1}(2w^{\cdot} - v\sin\theta/r)'\right] + n_{xy}\Big(u^{\cdot} - v\cos\theta/r \\
&- \frac{K}{R_1}w^{\cdot\prime}\Big) + pw/2\Bigg\}\delta u + \Bigg\{A_{13}\left[u' + w/R_1 - \frac{K\sin\theta}{r}(w' - u/R_1)'\right] + (A_{23}+n_{xy})\Big[u\cos\theta/r + v^{\cdot} \\
&+ w\sin\theta/r - \frac{K\sin\theta}{r}(w^{\cdot\cdot} + w'\cos\theta/r)\Big] + A_{33}\left[u^{\cdot} + v' - v\cos\theta/r - \frac{K\sin\theta}{r}(2w^{\cdot} - v\sin\theta/r)'\right] \\
&+ n_{xx}\left[v' - \frac{K\sin\theta}{r}(w^{\cdot} - v\sin\theta/r)'\right]\Bigg\}\delta v + \Bigg\{\Bigg[-\frac{1}{r}D_xKr(A_{11}+n_{xx})D_x \\
&+ A_{12}\frac{K\cos\theta}{r}D_x - 2D_yA_{13}KD_x + n_{xx} - \frac{1}{r}D_yn_{xy}KrD_x\Bigg](w' - u/R_1) \\
&+ \left[-\frac{1}{r}D_xA_{12}Kr + (A_{22}+n_{yy})K\cos\theta/r - D_y(2A_{23}+n_{xy})K\right]
\end{aligned}
$$

$$\times(w^{\cdot\cdot}+w'\cos\theta/r)+\left[-\frac{1}{r}D_xA_{13}KrD_x+A_{23}\frac{K}{r}\cos\theta\, D_x-2D_yA_{33}KD_x\right]$$

$$\times(2w^{\cdot}-v\sin\theta/r)+\left[-\frac{1}{r}D_yn_{xx}KrD_x+n_{xy}\left(1+\frac{K\cos\theta}{r}D_x\right)\right](w^{\cdot}-v\sin\theta/r)$$

$$\left.-\frac{1}{r}(D_xn_{xy}Kr+D_yn_{yy}Kr)w^{\cdot\prime}-pu/2\right\}\delta w+[(A_{11}+n_{xx})KD_x(w'-u/R_1)$$

$$\left.+A_{12}K(w^{\cdot\cdot}+w'\cos\theta/r)+A_{13}KD_x(2w^{\cdot}-v\sin\theta/r)+n_{xy}Kw^{\cdot\prime}]\delta w'\right)\mathrm{d}y=0 \tag{20}$$

under the presumption that uniform hydrostatic pressure p acts on the complete exterior shell surface. In (19) and (20) occasionally for clerical reasons D_x, D_y have been introduced as symbols for ∂/∂_x, ∂/∂_y, the convention being that these operate on all ensuing factors constituting a term and for instance $\partial^2w/\partial x\partial y$ is denoted by $D_{yx}w$ or alternatively $w^{\cdot\prime}$.

A crucial test at this stage is naturally an examination if any physically meaningful results may be obtained when interpreting the dynamical parts of the boundary condition (20) in terms of stress-rate resultants and stress couples. It is customary in the literature on elastic shells to derive resulting dynamic variables by aid of a virtual work principle based on middle surface strain and curvature change as kinematic variables. In the present three-dimensional approach however, no specific appeal has been made to separate but additive energy contributions of middle surface stretching and change of curvature and therefore, following an approach similar to the one out-lined by Sewell[23] for a corresponding plate problem, a reversed procedure is adopted by introducing tentatively the stress rate resultants

$$\dot{n}^{\alpha\beta}=\int_{-t/2}^{t/2}\dot{s}^{\alpha\beta}\,\mathrm{d}z \tag{21}$$

and stress couples

$$\dot{m}^{\alpha\beta}=\int_{-t/2}^{t/2}\epsilon^{\beta\delta}(\dot{s}^{\alpha_\delta}z+s^{\alpha_\delta}w)\,\mathrm{d}z \tag{22}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional permutation tensor.

Eliminating the transverse shear forces the conditions for translational balance then read in physical components as obtained from static equilibrium considerations

$$\left.\begin{aligned}
&\frac{\partial r\dot{n}_{xx}}{r\partial x}+\frac{\partial\dot{n}_{yx}}{\partial y}-\dot{n}_{yy}\cos\theta/r+\frac{1}{R_1}\left(\frac{\partial r\dot{m}_{xy}}{r\partial x}+\frac{\partial\dot{m}_{yy}}{\partial y}+\dot{m}_{yx}\cos\theta/r\right)+p\left(\frac{\partial w}{\partial x}-\frac{u}{R_1}\right)=0\\
&\frac{1}{r}\frac{\partial r\dot{n}_{xy}}{\partial x}+\frac{\partial\dot{n}_{yy}}{\partial y}+\dot{n}_{yx}\cos\theta/r-\frac{\sin\theta}{r}\left(\frac{\partial r\dot{m}_{xx}}{r\partial x}+\frac{\partial\dot{m}_{yx}}{\partial y}-\dot{m}_{yy}\cos\theta/r\right)+p\left(\frac{\partial w}{\partial y}-v\sin\theta/r\right)=0\\
&\dot{n}_{xx}/R_1+\dot{n}_{yy}\sin\theta/r-\frac{1}{r}\frac{\partial}{\partial x}\left(\frac{\partial r\dot{m}_{xy}}{\partial x}+r\frac{\partial\dot{m}_{yy}}{\partial y}+\dot{m}_{yx}\cos\theta\right)+\frac{1}{r}\frac{\partial}{\partial y}\left(\frac{\partial r\dot{m}_{xx}}{\partial x}+r\frac{\partial\dot{m}_{yx}}{\partial y}-\dot{m}_{yy}\cos\theta\right)\\
&\quad+p\left(\frac{\partial u}{\partial x}+u\cos\theta/r+\partial v/\partial y+w/R_1+w\sin\theta/r\right)=0
\end{aligned}\right\} \tag{23}$$

stressing once more the fact that the dependent variables derive from the difference between two solutions (and consequently any terms containing $\dot{p}$, which is prescribed, cancel in (23)).

By aid of (5), (16), (17), (21) and (22) the Euler equation $(19)_3$ may then be exactly recovered from $(23)_3$. As regards the remaining equations $(19)_1$ and $(19)_2$, these may be recovered from $(23)_1$ and $(23)_2$ respectively only through multiplication of the integrand in (21), when evaluating $\dot{n}_{yx}$ and $\dot{n}_{yy}$ to be inserted in $(19)_1$ by a factor $(1-z/R_1)$ and correspondingly a factor $(1-z/R_2)$ when it comes to $(19)_2$. These operations are however irrelevant in the present approximation.

The boundary condition (20) may then be interpreted to the same order of approximation as

$$\oint\left[(\dot{n}_{xx}-pw/2)\delta u+\dot{n}_{xy}\delta v+\left(\dot{q}_x-\frac{\partial\dot{m}_{xx}}{\partial y}-pu/2\right)\delta w+(\dot{n}_{xx}w-\dot{m}_{xy})\delta\left(\frac{\partial w}{\partial x}\right)\right]\mathrm{d}y=0 \tag{24}$$

when introducing a transverse shear stress resultant formally defined as

$$\dot{q}_x = \frac{\partial \dot{m}_{xy}}{\partial x} + \frac{\partial \dot{m}_{yy}}{\partial y} + (\dot{m}_{xy} + \dot{m}_{yx}) \cos \theta / r \tag{25}$$

It is self-evident from the basic assumptions introduced above that this shear force rate is not consistent with the utilized particle velocity field through the constitutive equation but indeed (25) is an equation for rotational balance derivable from static considerations and it should be further noticed that the stress-rate resultants and the pressure p are not independent in (24) but related basically through (2).

Summing up then, the Euler eqns (19) together with the boundary condition (20) in general constitute the basis of a physically consistent and mathematically well set two-dimensional eigenvalue problem generating the magnitudes of critical loads for shells which buckle in modes being at least approximately covered by the adopted class of particle velocities (17). There seems to be one exception though due to the awkward presence of the p-dependent terms in (24). Thus for the particular case of a shell subjected to hydrostatic pressure and with ends free to move in meridional and transverse directions, there is an unhappy mixture of dynamical and kinematical variables in the boundary conditions (24) if stretching effects are of importance.

As regards the magnitude of errors involved there is no justification to discuss those inherent in the general set of shell equations except for specific load distributions as concluded by Niordson[14] in an isotropic, elastic context. Under the present circumstances such a discussion must also include the role of the (anisotropic) material properties and this is not attempted in general here. Instead the intrinsic errors in the governing equations are discussed in relation to a specific problem viz a circular cylinder under axial compression. This case has been dealt with extensively for elastic materials.

A CIRCULAR CYLINDER UNDER AXIAL COMPRESSIVE LOAD

For a right cylinder shell of uniform sheet thickness t and middle surface transversal radius of curvature a, suffering under a current compressive stress σ, the degenerate general rate equations (19) may be arranged in the symmetric fashion

$$\left.\begin{aligned} L_{11}u + L_{12}v + L_{13}w &= 0 \\ L_{12}u + L_{22}v + L_{23}w &= 0 \\ L_{13}u + L_{23}v + L_{33}w &= 0 \end{aligned}\right\} \tag{26}$$

when introducing the operators

$$\begin{aligned} L_{11} &= (a_{11} - \sigma)D_{xx} + 2a_{13}D_{xy} + a_{33}D_{yy} \\ L_{12} &= a_{13}D_{xx} + (a_{12} + a_{33})D_{xy} + a_{23}D_{yy} \\ L_{13} &= \frac{1}{a}(a_{12}D_x + a_{23}D_y) \\ L_{22} &= (a_{33} - \sigma)(1 + K/a^2)D_{xx} + 2a_{23}D_{xy} + a_{22}D_{yy} \\ L_{23} &= \frac{1}{a}(a_{23}D_x + a_{22}D_y) - \frac{K}{a}[a_{13}D_{xxx} + (2a_{33} - \sigma)D_{xxy} + a_{23}D_{xyy}] \\ L_{33} &= a_{22}/a^2 + K[(a_{11} - \sigma)D_{xxxx} + 4a_{13}D_{xxxy} + (2a_{12} + 4a_{33} - \sigma) \\ &\qquad \times D_{xxyy} + 4a_{23}D_{xyyy} + a_{22}D_{yyyy}] + \sigma D_{xx}. \end{aligned}$$

The boundary condition (20) reduces to

$$\oint \left(\left\{ [(a_{11} - \sigma)D_x + a_{13}D_y]u + (a_{12}D_y + a_{13}D_x)v + \frac{a_{12}}{a}w \right\} \delta u \right.$$
$$+ \left\{ (a_{13}D_x + a_{33}D_y)u + [a_{23}D_y + (a_{33} - \sigma)(1 + K/a^2)D_x]v \right.$$

$$-\frac{K}{a}\left[a_{13}D_{xx}-a_{23}\left(\frac{1}{K}-D_{yy}\right)+(2a_{33}-\sigma)D_{xy}\right]w\Big\}\delta v$$

$$-K\Big\{-\frac{1}{a}[a_{13}D_{xx}+(2a_{33}-\sigma)D_{xy}]v+\Big[(a_{11}-\sigma)D_{xxx}+(a_{12}+4a_{33}-\sigma)D_{xyy}$$

$$+4a_{13}D_{xxy}+2a_{23}D_{yyy}+\frac{\sigma}{K}D_x\Big]w\Big\}\delta w+K\Big\{-\frac{1}{a}a_{13}D_xv$$

$$+[(a_{11}-\sigma)D_{xx}+a_{12}D_{yy}+2a_{13}D_{xy}]w\Big\}\delta w'\Big)\,dy=0. \tag{27}$$

An interesting comparison between critical loads predicted from various theories in case of isotropic elastic shell material has been recently offered by Dym[15, 16]. Except for some of the pre-stress terms in L_{33} in (26) these equations are very similar in their structure to those attributed to Flügge and Koiter–Budiansky. The difference actually derives from terms which depend on the smallness of t/a and by themselves differ in those two theories. Both these sets of equations were shown by Dym to be able to predict expected buckling behaviour also in the limiting cases of very short and very long shells.

To solve (26) and (27) for a general case, accounting for the influence of all constitutive and geometric parameters, seems to be a formidable task and also of dubious practical value as it is well known that for instance imperfections in the cylinder geometry are quite decisive when failure occurs in the elastic range. A facilitating procedure which linearizes the equations governing the search for the eigenvalues of (26), (27), would be to delete all σ-dependent terms in (26) except for the very last one in L_{33} together with all terms of the type differing in the Flügge and Koiter-Budiansky equations. The governing equations then reduce to what is known in the elastic context as the Donnell approximation. Performing such an operation is not without ambiguity though as it is known for instance that for long cylinders which buckle in a column mode the Donnell approach overestimates the buckling load by a factor of two. This error is mainly due to the deletion of the σ-dependent term in L_{22}. A deletion of the stress-dependent term in L_{11}, is of minor importance which is apparent from the structure of (18) bearing in mind the assumption of positive definiteness of the constitutive matrix. A similar conclusion has been drawn earlier by Koiter[17] in an elastic context.

However the Donnell approximation seems to be applicable for shells which buckle in shallow modes and especially for cylinders of certain length to radius ratios $0,1\le L/ma\le 10$, m being the axial wave number, it was found by Dym for some boundary conditions that all the competing theories indicated so far gave approximately the same result. In order to achieve some explicit results for different boundary conditions and constitutive properties in the present case, without too cumbersome algebra, it seems reasonable to conjecture that a corresponding Donnell approximation in the plastic range would yield relevant results at least for some particular shell geometries.

Introducing the Donnell approximation into (26) yields

$$\left.\begin{aligned}
&a_{11}\frac{\partial^2u}{\partial x^2}+a_{33}\frac{\partial^2u}{\partial y^2}+(a_{12}+a_{33})\frac{\partial^2v}{\partial x\partial y}+a_{12}\frac{1}{a}\frac{\partial w}{\partial x}=0\\
&(a_{12}+a_{33})\frac{\partial^2u}{\partial x\partial y}+a_{33}\frac{\partial^2v}{\partial x^2}+a_{22}\frac{\partial^2v}{\partial y^2}+a_{22}\frac{1}{a}\frac{\partial w}{\partial y}=0\\
&a_{12}\frac{1}{a}\frac{\partial u}{\partial x}+a_{22}\frac{1}{a}\frac{\partial v}{\partial y}+K\Big[a_{11}\frac{\partial^4w}{\partial x^4}+(2a_{12}+4a_{33})\frac{\partial^4w}{\partial x^2\partial y^2}\\
&\quad+a_{22}\frac{\partial^4w}{\partial y^4}\Big]+a_{22}\frac{w}{a^2}+\sigma\frac{\partial^2w}{\partial x^2}=0
\end{aligned}\right\} \tag{28}$$

when leaving out any terms accounting for interference between normal stresses and shear strains i.e. setting $a_{13}=a_{23}=0$ in (26).

The associated particular boundary conditions singled out for study reduce in this approximation to

$$a_{11}\frac{\partial u}{\partial x}+a_{12}\frac{\partial v}{\partial y}=0, \tag{29}$$

$$v=0 \tag{30a}$$

or alternatively

$$\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0, \tag{30b}$$

$$\frac{\partial^2 w}{\partial x^2}=0, \tag{31}$$

$$w=0. \tag{32}$$

Physically the dynamic boundary condition (29) implies that the axial load rate is controlled (in contrast to the conjugate displacement rate). The condition (30b) is a relaxed form of (30a) corresponding to in-plane shear stress free ends of the cylinder (in a practical situation the axial load is then imagined to be introduced through some shear weak member). This option of in-plane boundary condition is left open as it is well known from earlier shell studies and also from the three-dimensional theory of uniqueness of finite deformation of different solids that the possibility of bifurcations is very sensitive to the manner in which external loads are introduced. Finally (31) and (32) imply simply supported ends. It is evident, though, that for instance (32) is not consistent with the assumption of a continuously varying steady membrane state except for a material which undergoes no lateral contraction during uni-axial loading. It has been shown, however, by Almroth[18] that this approximation as regards the resulting buckling stress is negligible at least in a corresponding elastic situation.

Now from $(29)_1$ and $(29)_2$ the in-plane velocities may be solved as

$$Lu=-\frac{1}{a}\left(a_{12}\frac{\partial^3 w}{\partial x^3}-a_{22}\frac{\partial^3 w}{\partial x^2\partial y}\right) \tag{33}$$

$$Lv=-\frac{1}{a}\left[\frac{1}{a_{33}}(a_{11}a_{22}-a_{12}^2-a_{12}a_{33})\frac{\partial^3 w}{\partial x^2\partial y}+a_{22}\frac{\partial^3 w}{\partial y^3}\right] \tag{34}$$

where the operator L is defined by

$$L=a_{11}\frac{\partial^4}{\partial x^4}+\frac{1}{a_{33}}(a_{11}a_{22}-a_{12}^2-2a_{12}a_{33})\frac{\partial^4}{\partial x^2\partial y^2}+a_{22}\frac{\partial^4}{\partial y^4}$$

Introduction of this result into $(29)_3$ yields a single equation

$$L\left\{K\left[a_{11}\frac{\partial^4 w}{\partial x^4}+2(a_{12}+2a_{33})\frac{\partial^4 w}{\partial x^2\partial y^2}+a_{22}\frac{\partial^4 w}{\partial y^4}\right]+\frac{\sigma}{a^2}\frac{\partial^2 w}{\partial x^2}\right\}+\frac{1}{a^2}(a_{11}a_{22}-a_{12}^2)\frac{\partial^4 w}{\partial x^4}=0 \tag{35}$$

for the eigenvalue problem.

For elastic-plastic strain-hardening materials having incremental elastic cubic symmetry, referred to the introduced stress and strain measures, and a smooth yield function, the introduced moduli may in the case of plastic loading be expressed as

$$\left.\begin{aligned}
&a_{11}=(1+\kappa\mu_{22}^2+4\gamma\kappa\mu_{12}^2)E/d, \quad a_{12}=(\nu-\kappa\mu_{11}\mu_{22}+4\nu\gamma\kappa\mu_{12}^2)E/d,\\
&a_{13}=-2\gamma\kappa\mu_{12}(\mu_{11}+\nu\mu_{22})E/d, \quad a_{22}=(1+\kappa\mu_{11}^2+4\gamma\kappa\mu_{12}^2)E/d,\\
&a_{23}=-2\gamma\kappa\mu_{12}(\nu\mu_{11}+\mu_{22})E/d, \quad a_{33}=\gamma[1-\nu^2+\kappa(\mu_{11}^2+\mu_{22}^2+2\nu\mu_{11}\mu_{22})]E/d
\end{aligned}\right\} \tag{36}$$

where

$$\gamma=G/E, \quad \kappa=E/h$$

and

$$d = 1 - \nu^2 + \kappa[\mu_{11}^2 + \mu_{22}^2 + 2\nu\mu_{11}\mu_{22} + 2\gamma(1-\nu^2)\mu_{12}^2];$$

E, ν and G being elastic constants in customary notation.

The main reason for introducing three independent elastic constants is simply to retain the possibility to separately investigate the significance of the magnitude of the in-plane shear modulus as regards the resulting buckling load.

To simplify the notation it is convenient to introduce the dimensionless moduli

$$\alpha_{11} = 1 + \kappa\mu_{22}^2, \quad \alpha_{12} = \nu - \kappa\mu_{11}\mu_{22}, \quad \alpha_{22} = 1 + \kappa\mu_{11}^2, \quad \alpha_{33} = \mathrm{d}G/E, \tag{37}$$

as in effect μ_{12} has been set to zero already in (28), and the dimensionless variables

$$\bar{w} = w/a, \quad \xi = \lambda x/a, \quad \eta = \lambda y/a, \quad 2\beta = \lambda^2\sigma/E, \quad \lambda = (da^2/K)^{1/4} \tag{38}$$

The governing eqn (31) then simplifies to

$$L'L''\bar{w} + \frac{\partial^4\bar{w}}{\partial\xi^4} = 0 \tag{39}$$

where L' is the dimensionless form of L and

$$L'' = \alpha_{11}\frac{\partial^4}{\partial\xi^4} + 2(\alpha_{12} + 2\alpha_{33})\frac{\partial^4}{\partial\xi^2\partial\eta^2} + \alpha_{22}\frac{\partial^4}{\partial\xi^4} + 2\beta\frac{\partial^2}{\partial\xi^2}. \tag{40}$$

Following a procedure outlined by Hoff and Rehfield [19] for a corresponding elastic situation, $\bar{w}$ may be formally solved from (39) as

$$\bar{w} = -\left(\frac{\partial}{\partial\xi}\right)^{-4} L'L''\bar{w} \tag{41}$$

Introducing this result into the boundary conditions (29) through (32) then yields

$$\left\{\alpha_{11}\alpha_{12} - \left[\alpha_{11}\alpha_{22} - \frac{\alpha_{12}}{\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - \alpha_{12}\alpha_{33})\right]\left(\frac{\partial}{\partial\xi}\right)^{-2}\frac{\partial^2}{\partial\eta^2} + \alpha_{12}\alpha_{22}\left(\frac{\partial}{\partial\xi}\right)^{-4}\frac{\partial^4}{\partial\eta^4}\right\}L''\bar{w} = 0 \tag{42}$$

$$\left[\alpha_{22}\left(\frac{\partial}{\partial\xi}\right)^{-4}\frac{\partial^3}{\partial\eta^3} + \frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - \alpha_{12}\alpha_{33})\left(\frac{\partial}{\partial\xi}\right)^{-2}\frac{\partial}{\partial\xi}\right]L''\bar{w} = 0 \tag{43a}$$

$$\left(\frac{\partial}{\partial\xi}\right)^{-1}\frac{\partial}{\partial\eta}L''\bar{w} = 0 \tag{43b}$$

$$\frac{\partial^2\bar{w}}{\partial\xi^2} = 0 \tag{44}$$

$$\bar{w} = 0 \tag{45}$$

Now a solution to (39) is sought for in the form

$$\bar{w} = e^{p\xi + i\epsilon\eta} \tag{46}$$

where p is in general a complex number and $\epsilon(da^2/K)^{1/4}$ is the circumferential wave number by aid of (38).

The characteristic equation associated with (39) then becomes

$$\left[\alpha_{11}p^4 - \frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - 2\alpha_{12}\alpha_{33})\epsilon^2p^2 + \alpha_{22}\epsilon^4\right][\alpha_{11}p^4 - 2(\alpha_{12} + 2\alpha_{33})\epsilon^2p^2 + \alpha_{22}\epsilon^4 + 2\beta p^2] + p^4 = 0. \tag{47}$$

For the degenerate isotropic case the roots of this equation have been given by Nachbar[20], curiously enough without any obvious indication of the solution method.

It is possible though by introducing the polynomial

$$P=\alpha_{11}p^4-\frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22}-\alpha_{12}^2-2\alpha_{12}\alpha_{33})\epsilon^2p^2+\alpha_{22}\epsilon^4 \tag{48}$$

and the constant

$$b=\beta+\frac{1}{2\alpha_{33}}(\alpha_{11}\alpha_{22}-\alpha_{12}^2-4\alpha_{12}\alpha_{33}-4\alpha_{33}^2)\epsilon^2 \tag{49}$$

to rearrange (47) to yield

$$P^2+2bp^2P+p^4=0 \tag{50}$$

The formal solution for P is then

$$P=(-b\pm i\sqrt{[1-b^2]})p^2 \tag{51}$$

and it is evident from the appearance of P in (48) that all the eight roots may be explicitly found. It may be readily seen that there are two distinct roots related to either sign in (51) and the remaining roots may be found as the negative and complex conjugate values of these.

To facilitate further discussion the cylinder length is in a strict sense assumed to be semi-infinite. This means that only roots of (47) having negative real parts are of interest. From a practical view-point, however, the results may as well apply to cylinders of finite length and of such geometry that the interference between the ends is negligible. Incidentally it was found by Batterman[2] in his axisymmetric study that the length of the cylinder played a minor role as regards the resulting buckling load in case of free ends.

Denoting the solution to (39) by

$$\bar{w}=\sum_{j=1}^{4}A_j\,e^{p_j\xi+i\epsilon\eta} \tag{52}$$

and introducing the polynomials

$$Q_i=\alpha_{11}p_i^4-2(\alpha_{12}+2\alpha_{33})\epsilon^2p_i^2+\alpha_{22}\epsilon^4+2\beta p_i^2$$
$$i=1,2,3,4 \tag{53}$$

the boundary conditions (42) through (45) may be expressed as

$$\sum_{i=1}^{4}A_i\left\{\alpha_{11}\alpha_{12}p_i^4+\left[\alpha_{11}\alpha_{12}-\frac{\alpha_{12}}{\alpha_{33}}(\alpha_{11}\alpha_{22}-\alpha_{12}^2-\alpha_{12}\alpha_{33}\right]\epsilon^2p_i^2+\alpha_{12}\alpha_{22}\epsilon^4\right\}\frac{Q_i}{p_i^4}=0 \tag{54}$$

$$\sum_{i=1}^{4}A_i\left[\frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22}-\alpha_{12}^2-\alpha_{12}\alpha_{33})p_i^2-\alpha_{22}\epsilon^2\right]\frac{Q_i}{p_i^4}=0 \tag{55a}$$

$$\sum_{i=1}^{4}A_i\frac{Q_i}{p_i}=0 \tag{55b}$$

$$\sum_{i=1}^{4}A_ip_i^2=0 \tag{56}$$

$$\sum_{i=1}^{4}A_i=0. \tag{57}$$

From the definitions of P and Q, eqns (48) and (53) respectively, it follows that

$$Q = P + 2bp^2 \tag{58}$$

and consequently by aid of (51)

$$Q = (b \pm i\sqrt{[1-b^2]})p^2. \tag{59}$$

Introducing the notation

$$c = b + i\sqrt{(1-b^2)} \tag{60}$$

and denoting the four roots p_i of interest by q, r, $\bar{q}$, $\bar{r}$ where the bar denotes complex conjugation, then from (59)

$$Q_1 = cq^2, \quad Q_2 = cr^2, \quad Q_3 = \bar{c}\bar{q}^2, \quad Q_4 = \bar{c}\bar{r}^2. \tag{61}$$

By aid of (48), (51) and (60) the boundary conditions (54) through (57) reduce to

$$(A_1 + A_2)[-\alpha_{12}\bar{c} + (\alpha_{11}\alpha_{22} - \alpha_{12}^2)\epsilon^2]c + (A_3 + A_4)[-\alpha_{12}c + (\alpha_{11}\alpha_{22} - \alpha_{12}^2)\epsilon^2]\bar{c} = 0 \tag{62}$$

$$(A_1 + A_2 + A_3 + A_4)\frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - \alpha_{12}\alpha_{33}) - \alpha_{22}\epsilon^2\left(\frac{A_1c}{q^2} + \frac{A_2c}{r^2} + \frac{A_3\bar{c}}{\bar{q}^2} + \frac{A_4\bar{c}}{\bar{r}^2}\right) = 0 \tag{63a}$$

$$A_1cq + A_2cr + A_3\bar{c}\bar{q} + A_4\bar{c}\bar{r} = 0 \tag{63b}$$

$$A_1q^2 + A_2r^2 + A_3\bar{q}^2 + A_4\bar{r}^2 = 0 \tag{64}$$

$$A_1 + A_2 + A_3 + A_4 = 0 \tag{65}$$

Remembering that $c\bar{c} = 1$ from (60), the eigenvalue equations for the two different cases then readily follow as

$$|q^2 - r^2| Im\, c\, Im \frac{c}{q^2r^2} = 0 \tag{66a}$$

for the rigid variant of the boundary conditions (63a) and

$$|q^2 - r^2|\, Im\, \bar{c}\, Im\, [c(\bar{q} + \bar{r})] = 0 \tag{66b}$$

for the relaxed form (63b) provided $\epsilon \neq 0$ in both cases. In (66) Im denotes the imaginary part of the quantity following.

In a general case $q \neq r$ and as q and r are solutions to

$$\alpha_{11}p^4 - \left[\frac{1}{\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - 2\alpha_{12}\alpha_{33})\epsilon^2 - \bar{c}\right]p^2 + \alpha_{22}\epsilon^4 = 0 \tag{67}$$

from (48), (51) and (60), q^2r^2 is a real positive number if ϵ is separate from zero. Consequently the only solution to (66a) is

$$Im\, c = 0. \tag{68}$$

The corresponding eigenvalue is then from (49) and (60)

$$\beta = 1 - \frac{1}{2\alpha_{33}}(\alpha_{11}\alpha_{22} - \alpha_{12}^2 - 4\alpha_{12}\alpha_{33} - 4\alpha_{33}^2)\epsilon^2 \tag{69}$$

The buckling load thus derived decreases with the circumferential wave number if

$2\alpha_{33} < (\alpha_{11}\alpha_{22})^{1/2} - \alpha_{12}$, which in a purely elastic situation corresponds to the delicate condition $G < E/[2(1+\nu)]$. This is an interesting effect indicating the significance of the in-plane shear material properties. Remembering, however, the approximative character of the rate equations utilized to arrive at this result, no far-reaching conclusions are drawn at the present moment but the matter should be worthy of a more profound investigation based on the complete set of shell equations.

In case of isotropic elastic properties (69) yields

$$\beta = 1 + \frac{\kappa}{1+\nu}\left(\frac{1}{2} - 3\mu_{11}\mu_{22}\right)\epsilon^2 \tag{70}$$

where the deviatoric property $\mu_{kk} = 0$ has been utilized. As the last term in (70) then increases with the circumferential wave number it is clearly of interest to investigate the axisymmetric case which however must be treated on its own merits in the present derivation of eigenvalue equations. It is readily shown from (28) through (32) that for a cylinder of finite length the critical loads for the two cases coincide and that $\beta = 1$ corresponds to a pure sinusoidal buckling mode if a large number of buckles in the axial direction is anticipated. The lowest buckling load is then from (38)

$$\sigma_{cr} = \frac{Et}{\sqrt{(3)}a\{1-\nu^2+\kappa[\frac{1}{2}-(1-2\nu)\mu_{11}\mu_{22}]\}^{1/2}}. \tag{71}$$

This result has been obtained earlier from somewhat simpler arguments by Ariaratnam and Dubey[3], who also departed from Hill's variational principle in their analysis. These writers concluded from their final approximation that the critical load dependence of the circumferential wave number was a function of the ratio between Young's modulus and the tangent modulus in conflict with the result reached above. The complete dynamic boundary conditions imposed were not explicitly stated in their analysis but were perhaps meant to be analogous to the present ones. Moreover their treatment was formally applicable to shells of finite length and bearing in mind the approximations introduced above this detail is not pursued further.

Ariaratnam and Dubey systematically investigated the sensitivity of the critical load to the direction of the yield surface normal, relevant to an anisotropic solid (as given in the present eqn (71)). The dependence is significant and as may be seen from (71) this effect is closely connected with the prevailing value of Poisson's ratio and particularly when $\nu = \frac{1}{2}$ it vanishes formally. Obviously though no far-reaching conclusions may be drawn for the limiting incompressible case remembering the kinematical restrictions introduced above.

Except for the solution (69) relevant to the rigid form of the boundary conditions, the only solution to (66b) is

$$Im\ [c(\bar{q}+\bar{r})] = 0. \tag{72}$$

The exact solution to this equation may not be easily found. In order to arrive at an approximate solution it might be noticed that in a practical situation ϵ may be expected to be small compared to unity as by definition $\epsilon(da^2/K)^{1/4}$ must be an integer from (38). Under the additional assumption that $\kappa\epsilon^2 \ll 1$, after lengthy but straightforward calculations by aid of (48), (49), (51) and (60), (72) may be solved to first order to yield

$$\beta = \frac{1}{2}\left(1 + \left\{1 + \sqrt{[(1+\kappa\mu_{11}^2)(1+\kappa\mu_{22}^2)]} + \frac{\kappa}{1+\nu}[1-(5-\nu)\mu_{11}\mu_{22}]\right\}\epsilon^2\right) \tag{73}$$

when the elastic part of incremental deformation is assumed isotropic and provided that, as earlier indicated, $\beta \neq 1$ and $\epsilon \neq 0$.

For this case the magnitude of the critical load increases with ϵ but the lowest value amounts, e.g. in the case of linear strain-hardening, to only about one half of that found for the rigid boundary conditions as there is a remarkable insensitivity to the circumferential wave number in (73). To elucidate on this matter (73) may be rewritten as

$$\beta = \tfrac{1}{2}(1 + \delta n^2) \tag{74}$$

where n is an integer greater than one as a column buckling mode is out of the discussion for reasons indicated above.

In case of common metals, cylinders ordinarily fail in the plastic range if $a/t < 100$ (say). Adopting this ratio and typical values $\nu = 1/3$, $\mu_{11} = -2\mu_{22} = 2/\sqrt{6}$, then δ in (74), which increases monotonically with $\kappa = E/h$, still only slightly exceeds 0·01 if $\kappa = 10$. In this situation $\kappa\epsilon^2 < 0{\cdot}02$ for $n = 2$.

Quantitatively, however, this reduced buckling load may not be accepted immediately without reservations. It goes without saying that the errors involved in the approximations introduced when arriving at any solution must be checked by aid of the original unreduced equations. Furthermore in a real situation there always exist edge zones in which the character of the stress distribution may not be determined from a two-dimensional theory.

A remark pertinent to the present case of relaxed boundary conditions has been given by Koiter[17] who demonstrated by a simple argument (related to deletion of a σ-dependent term corresponding to the one in L_{22} in (26) in the present treatment) that the Donnell approximation may underestimate the critical load for short elastic shells ($L^2/(at) \ll 1$). It was shown in a later numerical study by Simmonds and Danielson[21] that except for very short and very long cylinders, the Donnell approximation generates accurate results for the buckling load in the elastic case. However as these writers remark this result might be fortuitous and attributable to the weak dependence of the buckling load on the circumferential wave number as is also clear from (73) above. It is evident though that the particular form of the boundary conditions is of considerable interest in the present context, as was also found by Batterman[2].

Before the solutions may be fully accepted it must be proved that the assumptions underlying the adoption of a linear comparison material are fulfilled implying a stable bifurcation. That this is indeed the case for the present boundary conditions is clear immediately as the amplitudes in the eigensolutions found may always be chosen small enough that loading in a plastic sense prevails everywhere in the cylinder when these eigenfields are superposed on the steady state homogeneous axial compression mode.

The rigid-plastic solution may not be recovered from the present analysis but this case must be treated on its own merits. This is due to the fact that when the ratio $E/h \to \infty$ only the restricted class of velocity fields compatible with the flow rule is admissible and the validity of the variational principle employed above is ruled out as the stress-rates are not derivable from a potential any longer. Evidently overlooking this restriction in a similar investigation of a cylinder subject to torsion, Neale[22] found a finite critical load in the rigid-plastic case although it was shown by Hill[23] that the deformation mode of a rigid-plastic bar in torsion is always unique.

For the present case the general velocity field admissible under a uniaxial stress state has been given by Prager[24] in case of an isotropic smooth yield function. In the present notation this field yields for the middle surface displacements

$$\left.\begin{aligned} u &= Axa\cos\phi + Bxa\sin\phi + C(a^2 + 2x^2) + Dx \\ v &= -\tfrac{1}{4}(a^2 - 2x^2)(A\sin\phi - B\cos\phi) \\ w &= -\tfrac{1}{4}(a^2 + 2x^2)(A\cos\phi + B\sin\phi) - 2Cxa - \frac{D}{2}a \end{aligned}\right\} \tag{75}$$

where A, B, C and D are mode amplitudes.

As may be seen from (75) this field admits only column buckling types of modes of no shear distorsion and the boundary conditions may only be approximately fulfilled. Suffice it to say in this connection that an analysis based on Prager's field has been carried out by Hill[23] for a rigid-plastic column.

CONCLUSION

Despite the merits of the fundamental principles laid down by Hill for elastic-plastic solids, they seem to have been utilized only to a small extent in the solution of problems of engineering significance. The generation of equations valid for idealized one- or two-dimensional cases must by necessity be an approximate procedure which is a common feature to any approach though. Some of the difficulties which may be encountered have been dwelt upon above and they have proved to be mostly of a technical nature leaving a minimum of assumptions to intuition once a class of particle velocity fields has been chosen. It seems quite plausible, in the present context, however, that still more refined shell equations may be derived by a more exhaustive utilization of the basic results won in modern elastic shell theory in combination with introduction of the idea of an elastic-plastic linear comparison solid. At the present moment a bifurcation theory for two-dimensional continua comparable in rigour and generality to Hill's three-dimensional theory, seems to be lacking.

Even for the very simple geometry and loading conditions prevailing in the illustration dealt with above computational difficulties were met with in the solution procedure. Under more complicated circumstances when even an analytical solution to the steady state may not be hoped for and numerical methods must be introduced from the very start, the access to a rigorous variational principle is of great value. For instance in shell problems, when bending stresses are of significance, the relevance of the common kinematical assumptions remains in severe doubt when it comes to strain-hardening solids and a three-dimensional approach may be necessary particularly when transverse shear effects are of importance. Tentatively such problems may be treated with success through a finite element method based on the variational principle and employing elements which may be expected to be in an approximately homogenous hardening state under the specific circumstances prevailing. Some of the main features appearing in such an approach are neatly exposed in a recent study by Needleman [25] of thick-walled spherical shells subject to internal pressure.

Whether the class of particle velocity fields commonly dealt with in elastic buckling theory, will be applicable also in elastic-plastic situations will to a high degree depend on the particular shell material properties. To render error estimates possible the access to three-dimensional analyses are necessary but very few such are available. The matter may be illustrated by some results of Strifors and Storåkers [26, 27], who have analysed, from a rigid-plastic point of view, two specific problems which fall in the present class. For a thick-walled circular cylinder under external hydrostatic pressure (when the particle velocity field generated by the flow rule is quite versatile) these writers found that a simple tangent modulus formula applied in the thin-shell limit while for a corresponding spherical case buckling was ruled out by rigid-plastic theory when a non-singular yield function was adopted. This latter unrealistic finding is due to the prevailing infinite modulus for plastic incremental shear strain referred to the steady state principal stress directions. The role of the in-plane shear modulus was shown to be of significance also in the cylinder problem dealt with above and it is well-known that the same state of affairs prevails when dealing with buckling of plates. Thus a proper choice of constitutive equation is of paramount importance. The diplomatic option of shear modulus in Bushnell's earlier mentioned problem setting seems very sensible especially when seen in the light of Budiansky's [28] discussion of the physical soundness of deformation theories.

The discussion of plastic plate buckling has centered very much around the question whether a corner develops on the material yield surface or not. An impression of confusion remains, though, as to why the application of non-classical constitutive theories should be particularly successful in a field where other kinds of second-order effects are of fundamental importance. As remarked by Phillips [29] this issue might not be crucial as a small initial curvature of a plate will cause deviations from idealized strain-paths under loading. As a consequence drastic changes may occur in the incremental properties for solids which develop pointed yield surfaces. The same effect may be caused also by excentric loading and the arguments apply to shells as well.

Accounting for these constitutive aspects and the role of the boundary conditions as dwelt upon above, might still not suffice to explain the notorious and embarrassing discrepancies between obtained theoretical and experimental results for buckling of shells in a plastic fundamental state. The influence of boundary constraints on the steady stress state is an important aspect which ought to be kept in mind when comparing experimental results and theoretical results based on mathematical models which only artificially simulate conditions prevailing in a laboratory test. In a numerical study of buckling of cylindrical sandwich shells, having stress-strain characteristics pertinent to an aluminium alloy, Murphy and Lee [30] found that bending waves causing plastic unloading developed during steady global compression. The determined collapse loads (maximum loads) were in good agreement with experimental results at least for axisymmetric buckling modes. Incidentally this proved to be the case also for bifurcation loads associated with cylinder ends free in Batterman's sense.

Furthermore in a real situation such features as presence of residual stresses and geometrical non-linearities introduced through imperfections may be highly decisive and a buckling value theoretically estimated from a bifurcation analysis may not be of much practical value. Bifurcations in elastic-plastic solids will mostly occur under increasing external loading and consequently in such cases the systems are not sensitive to geometric imperfections in the sense common in the theory of elasticity. Despite this formal insensitivity when plastic effects are significant, initial imperfections might as well be the latent major cause of failure. This conclusion may be drawn especially from a recent study by Hutchinson[31].

For two simple systems, a compressed idealized column with a non-linear spring support and an externally pressurized spherical shell, which are both imperfection-sensitive in the elastic range, Hutchinson showed that the increase in load-carrying capacity, after a bifurcation has occurred during the deformation of a perfect system, is not significant in the plastic range. When initial imperfections are present the load-carrying capacity is substantially reduced as for instance in the shell problem, an initial imperfection in the middle surface geometry of around 0·4 times the shell thickness, lowers the maximum load by a factor of two for particular but realistic shell parameters. In an approximate, though apparently quite accurate, study Neale[32] found similar features for cylindrical shells under torsion.

Thus it would seem to be rewarding and also necessary to investigate if such circumstances prevail e.g. for the cylinder problem discussed above before making any close comparison between theoretical and experimental results. For a particular case some insight is already available from Murphy and Lee's[30] discussion of a sandwich shell of initial geometry characterized by imperfections in a diamond shape pattern. As remarked by Hutchinson[33], in practice there might also exist situations when, due to the presence of geometric imperfections, real cylinders buckle undergoing purely elastic deformation when bifurcations would be expected in the plastic range only in the case of perfect geometry.

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